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Method to determine cutoff frequencies for acoustic waves propagating in nonisothermal media

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A method to determine cutoff frequencies for linear acoustic waves propagating in nonisothermal media is introduced. The developed method is based on wave variable transformations that lead to Klein-Gordon equations, and the oscillation theorem is applied to obtain the turning point frequencies. Physical arguments are used to justify the choice of the largest turning point frequency as the cutoff frequency. The method is used to derive the cutoff frequencies in nonisothermal media modeled by exponential and power law temperature gradients, for which the cutoffs cannot be obtained based on known analytical solutions. An interesting result is that the acoustic cutoff frequencies calculated by the method are local quantities that vary in the media, and that their specific values at a given height determine the frequency that acoustic waves must have in order to be propagating at this height. To extend this physical interpretation of the acoustic cutoff frequency to nonisothermal media of arbitrary temperature gradients, a generalized version of the method applicable to these media is also presented.

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I. INTRODUCTION

Theories of propagation of acoustic waves in homogeneous media are based on a dispersion relation derived from the acoustic wave equation, which is solved by making Fourier transforms in time and space. This well-known approach has also been used to study the wave propagation in inhomogeneous media [1], however, the resulting dispersion relation is only valid locally where the wavelength is shorter than the characteristic scales over which the basic physical parameters vary [2–4]. There are some physical situations where a dispersion relation can be derived for an inhomogeneous medium in which gradients of the physical parameters do not directly affect the speed of sound [2,4–7]. Among these cases, probably the best known is the one originally studied by Lamb [8].

The background medium considered by Lamb [8] is isothermal and stratified; this medium is also called "*isothermal atmosphere*" because of its applications to solar and stellar atmospheres [6]. Lamb showed that the propagation of acoustic waves in this medium is determined by the acoustic cutoff frequency, which is defined as the ratio of the speed of sound to twice the density (or pressure) scale height. The physical meaning of this cutoff is that the wave propagation is affected by the density gradient only when the wavelength is equal to, or longer than, the density scale height. Otherwise, the waves propagate freely in the medium because the cutoff frequency is global (the same in the entire medium) and, therefore, its effect on the wave propagation is the same at each atmospheric height (see Appendix A for further explanation).

Lamb [8] also demonstrated that the waves are propagating only when their frequencies are higher than the cutoff, otherwise they are evanescent, and that the cutoff is the natural frequency of the atmosphere [9]; the latter simply means that any acoustic disturbance imposed on the atmosphere would trigger an atmospheric response at the cutoff frequency [10]. Lamb's results have been widely used in studies of acoustic waves ranging from some laboratory experiments [2,4] and Earth's free atmospheric oscillations [11] to planetary free oscillations [12], especially, acoustic oscillations of Jupiter [13], the characteristic 5-min and 3-min free oscillations of the Sun and its atmosphere [14,15], and stellar oscillations [16]. Since temperature gradients play a dominate role in all these settings, the use of Lamb's cutoff and natural frequencies must only be considered as a simple approximation. Clearly, a method that would allow extending Lamb's results to nonisothermal media is needed. Such a method is developed in this paper.

In the past, the propagation of linear acoustic waves in nonisothermal was studied by using the so-called WKB approach [2–4,17], which is only valid for waves with wavelengths shorter than the characteristic scales over which the physical parameters vary. In addition some authors have treated Lamb's global cutoff frequency as a local quantity in a nonisothermal medium and calculated its variation with position in the background medium [14]. Clearly, this treatment cannot be justified in most physical situations for which different approaches are required [18,19].

In this paper, our approach is to study the propagation of linear and adiabatic acoustic waves in nonisothermal media, which are modeled with exponential and power law temperature distributions. For each model, we derived the acoustic wave equation and found its analytical solutions. Then, we tried to determine the acoustic cutoff frequency for each model by using the solutions. Unfortunately, this was only possible for one special model. Hence, we developed a method that is based on wave variable transformations that lead to Klein-Gordon equations. The method uses the oscillation theorem to determine the turning point frequencies and selects the largest of these frequencies as the cutoff frequency. We applied the method to our models of nonisothermal media and derived the cutoff frequencies. The method was also generalized so that it can be applied to nonisothermal media of arbitrary temperature gradients.

The outline of this paper is as follows. The acoustic wave equations for our models of nonisothermal media are derived in Sec. II. Determination of the acoustic cutoff frequencies by using steady-state analytical solutions is described in Sec. III. Development of our method and its applications to the considered models are presented in Sec. IV. Our generalized method, applicable to nonisothermal media of arbitrary temperature gradients, is described in Sec. V. A brief summary of our results is given in Sec. VI.

II. ACOUSTIC WAVE EQUATIONS IN NONISOTHERMAL MEDIA

We consider a nonisothermal medium in which the temperature T_0 and density ρ_0 vary only in one direction z in such a way that the gas pressure, defined as $p_0 = RT_0(z)\rho_0(z)/\mu$, where R is the universal gas constant and μ is the mean molecular weight, remains constant and therefore gravity can be neglected. For simplicity, we assume that acoustic waves are linear and adiabatic and that they propagate solely in the z direction. The waves are described by the following variables: velocity $u_1(t,z)$, pressure $p_1(t,z)$, and density $\rho_1(t,z)$ perturbations [19]. Applying these assumptions to the standard set of linearized hydrodynamic equations [20], we write the continuity, momentum and energy equations as

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial(\rho_0 u_1)}{\partial z} = 0, \tag{1}$$

$$\rho_0 \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial z} = 0, \tag{2}$$

$$\frac{\partial p_1}{\partial t} + u\frac{dp_0}{dz} - c_s^2\left(\frac{\partial \rho_1}{\partial t} + u\frac{d\rho_0}{dz}\right) = 0, \tag{3}$$

where the speed of sound is $c_s = [\gamma p_0/\rho_0(z)]^{1/2} = [\gamma R T_0(z)/\mu]^{1/2}$, with γ being the ratio of specific heats. The energy equation can be further simplified by using $p_0 = \text{const}$ and combining Eq. (3) with Eq. (1). This gives

$$\frac{\partial p_1}{\partial t} + \rho_0 c_s^2 \frac{\partial u_1}{\partial z} = 0, \tag{4}$$

where $\rho_o(z)c_s^2(z) = \text{const}$ in all nonisothermal models considered in this paper.

Using Eqs. (1), (2), and (4), we derive the following acoustic wave equations:

$$\frac{\partial^2 u_1}{\partial t^2} - c_s^2(z)\frac{\partial^2 u_1}{\partial z^2} = 0, \tag{5}$$

$$\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial}{\partial z}\left[c_s^2(z)\frac{\partial p_1}{\partial z}\right] = 0 \tag{6}$$

and

$$\frac{\partial^2 \rho_2}{\partial t^2} - \frac{\partial^2}{\partial z^2}[c_s^2(z)\rho_2] = 0, \tag{7}$$

where the new wave variable ρ_2 is related to ρ_1 through $\partial\rho_2/\partial z = \rho_1$. Since the wave equations have different form for each wave variable, all of them must be included in our analysis [19]. This is an interesting set of wave equations for which the wave variable u_1 depends only on $c_s(z)$, p_1 depends on both $c_s(z)$ and its first derivative, and ρ_2 depends on $c_s(z)$ as well as on its first and second derivatives. Note that this set of equations is valid only when other external forces such as gravity and magnetic field do not have any influence on wave propagation [4].

To introduce the models of nonisothermal media, we define $\xi = z/z_0$, where z_0 is a given fixed height, and use the exponential and power law temperature distributions of $T_0(\xi)$ in the background medium. In the exponential model, $T_0(\xi) = T_{00}e^{\xi}, \rho_0(\xi) = \rho_{00}e^{-\xi}$, and $c_s(\xi) = c_{s0}e^{\xi/2}$, where T_{00}, ρ_{00}, and c_{s0} represent the temperature, density, and speed of sound at the height z_0, respectively. We also consider an infinite series of power law models with $T_0(\xi) = T_{00}\xi^m, \rho_0(\xi) = \rho_{00}\xi^{-m}$, and $c_s(\xi) = c_{s0}\xi^{m/2}$, where m is a positive integer. The variations of the speed of sound in the exponential and power law models are shown in Fig. 1.

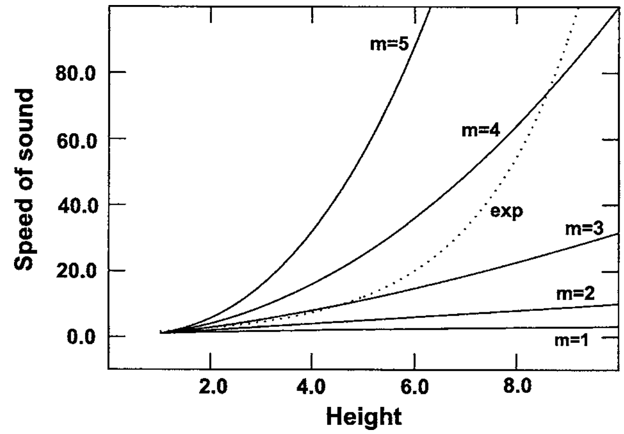

FIG. 1. The speed of sound c_s normalized by c_{s0} is plotted as a function of height z/z_0 for the exponential model (exp) and the power law models with $m=1$, 2, 3, 4, and 5; note that all the models start at $z=z_0$.

III. ANALYTICAL SOLUTIONS AND DETERMINATION OF CUTOFF FREQUENCIES

To obtain analytical solutions of the acoustic wave equations [see Eqs. (5)–(7)], we first derive the steady-state wave equations by making Fourier transforms in time $[u_1(t,\xi), p_1(t,\xi), -\rho_2(t,\xi)] = [u_2(\xi), p_2(\xi), \rho_3(\xi)]e^{-i\omega t}$, with ω being the wave frequency. Then, the steady-state solutions are obtained for each model separately. The solutions are presented by using the following notation: $C_{1,2,3}^{u,p,\rho}$, which are the integration constants, $\eta_0 = \omega z_0/c_{s0}$ and $2\eta_0 = \omega/\Omega_0$, with $\Omega_0 = c_{s0}/2z_0$, and J and Y are Bessel and Weber functions, respectively.

A. Exponential model

In this model, $c_s(\xi) = c_{s0}e^{\xi/2}$ and the steady-state wave equations can be written as

$$\frac{d^2u_2}{d\xi^2} + \eta_0^2 e^{-\xi} u_2 = 0, \quad (8)$$

$$\frac{d^2p_2}{d\xi^2} + \frac{dp_2}{d\xi} + \eta_0^2 e^{-\xi} p_2 = 0, \quad (9)$$

$$\frac{d^2\rho_3}{d\xi^2} + 2\frac{d\rho_3}{d\xi} + (1 + \eta_0^2 e^{-\xi})\rho_3 = 0. \quad (10)$$

Steady-state solutions of these equations can be easily found after the equations are transformed into the corresponding Bessel equations [21]. This gives

$$u_2(\xi) = C_1^u J_0(2\eta_0 e^{-\xi/2}) + C_2^u Y_0(2\eta_0 e^{-\xi/2}), \quad (11)$$

$$p_2(\xi) = e^{-\xi/2}[C_1^p J_1(2\eta_0 e^{-\xi/2}) + C_2^p Y_1(2\eta_0 e^{-\xi/2})], \quad (12)$$

and $\rho_3(\xi)=e^{-\xi}u_2(\xi)$. Based on the properties of Weber functions [22], we have $Y_0 \to \infty$ and $Y_1 \to \infty$ as $e^{-\xi/2} \to 0$ or $\xi \to \infty$. Hence, to obtain the finite solutions, we assume that $C_2^u=0$ and $C_2^p=0$.

B. Power law models

In these models, $c_s(\xi)=c_{s0}\xi^{m/2}$ and the steady-state wave equations are

$$\xi^2\frac{d^2u_2}{d\xi^2} + \eta_0^2 \xi^{2-m} u_2 = 0, \quad (13)$$

$$\xi^2\frac{d^2p_2}{d\xi^2} + m\xi\frac{dp_2}{d\xi} + \eta_0^2 \xi^{2-m} p_2 = 0, \quad (14)$$

$$\xi^2\frac{d^2\rho_3}{d\xi^2} + 2m\xi\frac{d\rho_3}{d\xi} + [m(m-1) + \eta_0^2 \xi^{2-m}]\rho_3 = 0. \quad (15)$$

To obtain the steady-state solutions, we transform these wave equations into their Bessel's forms [21]. For convenience, we present the solutions for the models with $m=0,1,2,3$, and 4 separately from those obtained for $m>4$.

1. Model with m=0

This is the simplest model as it describes a homogeneous medium with $c_s=c_{s0}$=const. In this case, Eqs. (13)–(15) have the same form for each wave variable and the solutions representing freely propagating acoustic waves are given by

$$[u_2(\xi),p_2(\xi),\rho_3(\xi)] = [C_1^u, C_1^p, C_1^\rho]\sin(\eta_0\xi) + [C_2^u, C_2^p, C_2^\rho]\cos(\eta_0\xi). \quad (16)$$

Since there are no gradients of physical parameters in this model, no cutoff frequency is present.

2. Model with m=1

In this model, $c_s(\xi)=c_{s0}\xi^{1/2}$ and the steady-state solutions can be written as

$$u_2(\xi) = \xi^{1/2}[C_1^u J_1(2\eta_0\xi^{1/2}) + C_2^u Y_1(2\eta_0\xi^{1/2})], \quad (17)$$

$$p_2(\xi) = C_1^p J_0(2\eta_0\xi^{1/2}) + C_2^p Y_0(2\eta_0\xi^{1/2}), \quad (18)$$

and $\rho_3(\xi)=\xi^{-1}u_2(\xi)$. Since both $Y_0 \to \infty$ and $Y_1 \to \infty$ when $\xi \to 0$, we take $C_2^u=0$ and $C_2^p=0$. Note that there are similarities between these solutions and those obtained for the exponential model with $c_s(\xi)=c_{s0}e^{\xi/2}$, as in both cases the solutions are given by the Bessel functions J_0 and J_1.

3. Model with m=2

This is an interesting case because $c_s(\xi)=c_{s0}\xi$ and the steady-state wave equation for u_2 [Eq. (13)] can be written in the form of Euler's equation [21]

$$\frac{d^2u_2}{d\xi^2} + \frac{C_0}{4\xi^2}u_2 = 0, \quad (19)$$

where $C_0=4\eta_0^2=\omega^2/\Omega_0^2$. It is well known that this equation has periodic solutions when $C_0>1$ or $\omega>\Omega_0$, nonperiodic solutions when $C_0<1$ or $\omega<\Omega_0$, and a "turning point" when $C_0=1$ or $\omega=\Omega_0$ [21]. In general, the solutions can be written as $u_2(\xi)=C_1^u e^{k_1 \ln \xi}+C_2^u e^{k_2 \ln \xi}$, where $k_{1,2}=1/2\pm i\alpha$ and $\alpha=(\omega^2-\Omega_0^2)^{1/2}/2\Omega_0$.

The wave equations for p_2 [Eq. (14)] and ρ_3 [Eq. (15)] are of different forms than Eq. (13) and yet their solutions can also be classified as the periodic ($\omega>\Omega_0$), nonperiodic ($\omega<\Omega_0$), and turning point ($\omega=\Omega_0$) solutions [21]. The general form of the solution for p_2 is $p_2(\xi)=C_1^p\xi^{k_1}+C_2^p\xi^{k_2}$ with $k_{1,2}=-1/2\pm i\alpha$, and for ρ_2 is $\rho_2(\xi)=C_1^\rho\xi^{k_1}+C_2^\rho\xi^{k_2}$ with $k_{1,2}=3/2\pm i\alpha$.

These results imply that the frequency $\Omega_0=c_{s0}/2z_0$ is the acoustic cutoff frequency Ω_{ac} for the waves propagating in the medium modeled with $m=2$, so $\Omega_{ac}=\Omega_0$. This is an interesting result as Ω_0 depends on the choice of z_0 and c_{s0}, but its value does not directly depend on the gradients of temperature and density in the medium. Note that the global dispersion relation cannot be obtained for this model because the steady-state wave equations have nonconstant coefficients.

4. Model with m=3

The steady-state solutions for this model, in which $c_s(\xi)=c_{s0}\xi^{3/2}$, are given in the following form:

$$u_2(\xi) = \xi^{1/2}[C_1^u J_{-1}(2\eta_0\xi^{-1/2}) + C_2^u Y_{-1}(2\eta_0\xi^{-1/2})], \quad (20)$$

$$p_2(\xi) = \xi^{-1}[C_1^p J_2(2\eta_0\xi^{-1/2}) + C_2^p Y_2(2\eta_0\xi^{-1/2})], \quad (21)$$

and $\rho_3(\xi)=\xi^{-3}u_2(\xi)$. Again, $Y_{-1}\to\infty$ and $Y_2\to\infty$ when $\xi\to 0$, so the finite solutions are obtained by taking $C_2^u=0$ and $C_2^p=0$. In addition, $J_{-1}=-J_1$.

5. Model with m=4

Another interesting case is the model with $m=4$ for which $c_s(\xi)=c_{s0}\xi^2$. The steady-state solutions are given by

$$u_2(\xi) = \xi^{1/2}[C_1^u J_{1/2}(\eta_0\xi^{-1}) + C_2^u J_{-1/2}(\eta_0\xi^{-1})], \quad (22)$$

$$p_2(\xi) = \xi^{-3/2}[C_1^p J_{3/2}(\eta_0\xi^{-1}) + C_2^p J_{-3/2}(\eta_0\xi^{-1})], \quad (23)$$

and $\rho_3(\xi)=\xi^{-4}u_2(\xi)$.

Since the Bessel functions $J_{\pm 1/2}$ and $J_{\pm 3/2}$ can be expressed by the trigonometric functions sine and cosine [22], we write

$$u_2(\xi) = \xi[\tilde{C}_1^u \sin(\eta_0 \xi^{-1}) + \tilde{C}_2^u \cos(\eta_0 \xi^{-1})], \tag{24}$$

where $\tilde{C}_1^u = C_1^u (2/\pi\eta_0)^{1/2}$ and $\tilde{C}_2^u = C_2^u (2/\pi\eta_0)^{1/2}$, and $p_2(\xi) = \xi^{-1/2} u_2(\xi)$ with $\tilde{C}_1^u$ and $\tilde{C}_2^u$ being replaced by $\tilde{C}_1^p = C_1^p(\eta_0)^{-3/2}$ and $\tilde{C}_2^p = C_2^p(\eta_0)^{-3/2}$, respectively.

These solutions represent propagating waves, however, comparing them to the propagating wave solutions given by Eq. (16), one sees that there is some limitation on their validity. This limitation is related to the fact that we consider acoustic waves propagating in the direction of increasing ξ (or z) and, as a result, the argument $(\eta_0 \xi^{-1})$ quickly approaches zero. Thus, we approximate the solutions by writing them as $u_2(\xi) \approx \tilde{C}_1^u \eta_0 + \tilde{C}_2^u \xi$ and $p_2(\xi) \approx \tilde{C}_1^p \eta_0 \xi^{-1/2} + \tilde{C}_2^p \xi^{1/2}$, which are valid when $\xi \to \infty$. Obviously, in this limit the solutions are no longer periodic wave solutions.

6. Model with $m > 4$

The general steady-state solutions for these models are

$$u_2(\xi) = \xi^{1/2}[C_1^u J_{1/(m-2)}(\beta_m \xi^{1-m/2}) + C_2^u J_{-1/(m-2)}(\beta_m \xi^{1-m/2})], \tag{25}$$

$$p_2(\xi) = \xi^{(1-m)/2}[C_1^p J_{(m-1)/(m-2)}(\beta_m \xi^{1-m/2}) + C_2^p J_{-(m-1)/(m-2)}(\beta_m \xi^{1-m/2})], \tag{26}$$

and $\rho_3(\xi) = \xi^{-m} u_2(\xi)$, with $\beta_m = 2\eta_0/(m-2)$. For $m = 4,5,6,7,8,\ldots$, the solutions for $u_2(\xi)$ are given, respectively, in terms of the Bessel functions [23] $J_{\pm 1/2}, J_{\pm 1/3}, J_{\pm 1/4}, J_{\pm 1/5}, J_{\pm 1/6}, \ldots$, and the solutions for $p_2(\xi)$ are given, respectively, in terms of the Bessel functions [23] $J_{\pm 3/2}, J_{\pm 4/3}, J_{\pm 5/4}, J_{\pm 6/5}, J_{\pm 7/6}, \ldots$.

C. Cutoff frequency

The obtained results clearly show that the analytical solutions can be obtained for the exponential model and the power law models with any value of m. In the case of the model with $m=2$, the analytical solution allowed us to determine the acoustic cutoff frequency $\Omega_{ac} = \Omega_0$, where $\Omega_0 = c_{s0}/2z_0$. We found that acoustic waves are propagating when their frequency $\omega > \Omega_{ac}$, and they are evanescent when $\omega \leqslant \Omega_{ac}$. We now explore whether the frequency Ω_0 plays the same role in the other models of nonisothermal media.

Let us begin our discussion with the nonisothermal power law models and consider first the model with $m=1$. Since $2\eta_0 = \omega/\Omega_0$, we plot the solutions of $u_2(\xi)$ and $p_2(\xi)$ [see Eqs. (17) and (18)] in Fig. 2 for three different cases $\omega > \Omega_0, \omega = \Omega_0$, and $\omega < \Omega_0$. Note that $\xi = z/z_0$ and that in all calculations described in this subsection the wave propagation starts at $\xi = 1$ or $z = z_0$. The results presented in Fig. 2 are obtained for $\omega = 2\Omega_0$ (solid line), $\omega = \Omega_0$ (dashed line), and $\omega = \Omega_0/2$(dotted line), and they show that in all cases the solutions represent propagating wave solutions. This implies that Ω_0 is not the cutoff frequency for the model.

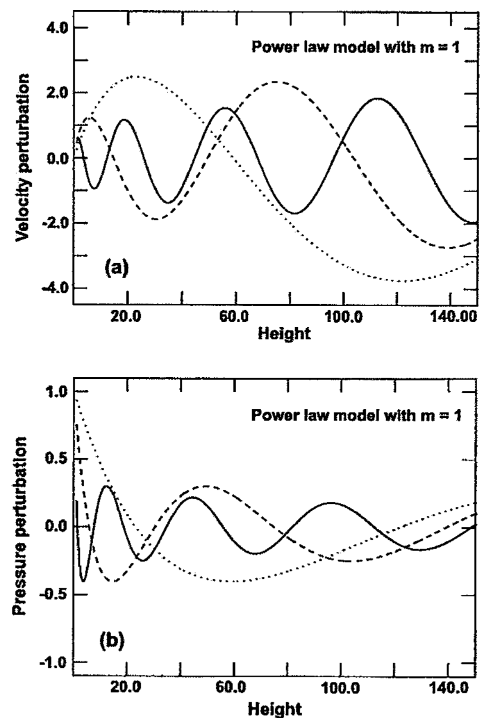

FIG. 2. Normalized velocity (a) and pressure (b) perturbations are plotted versus height in the nonisothermal model with the power law temperature distribution with $m=1$. The three considered cases correspond to $\omega = 2\Omega_0$ (solid line), $\omega = \Omega_0$ (dashed line), and $\omega = \Omega_0/2$ (dotted line). In all these cases, the wave propagation begins at the height $z = z_0$.

We now consider the model with $m=3$ and plot the solutions of $u_2(\xi)$ and $p_2(\xi)$, see Eqs. (20) and (21), in Fig. 3 for three different cases: $\omega = 50\Omega_0$ (solid line), $\omega = 25\Omega_0$ (dashed line), and $\omega - 10\Omega_0$ (dotted line). It is seen that for each case there is a height in the medium at which the waves become nonpropagating; the higher the frequency the larger the height. By comparing these results to those obtained for the model with $m=2$, we conclude that Ω_0 cannot be the cutoff frequency for the model with $m=3$. It is easy to show that this conclusion is also valid for all the models with $m \geqslant 4$.

There are some similarities between the solutions obtained for the exponential model and the power law model with $m=1$ as in both cases the solutions are given by the Bessel functions J_0 and J_1. Despite these similarities, the behavior of the waves in both models is completely different (compare Figs. 2 and 4). In the model with $m=1$, the waves are propagating, however, the same waves become quickly nonpropagating in the exponential model. In addition, the velocity $u_2(\xi)$ and pressure $p_2(\xi)$ perturbations [see Eqs. (16) and (17)] plotted as a function of height in Fig. 4 show that the frequency Ω_0 cannot be the acoustic cutoff for this model. The reason is that acoustic waves with $\omega = 50\Omega_0$ are already nonpropagating at the height $z/z_0 = 5$.

It is clear that the obtained steady-state solutions do not allow us to determine the acoustic cutoff frequencies for the nonisothermal models considered herein. The exception is the model with $m=2$ for which the frequency Ω_0 becomes the acoustic cutoff frequency $\Omega_{ac} = \Omega_0$. The role played by Ω_0 in this model is confirmed by the properties of the steady-

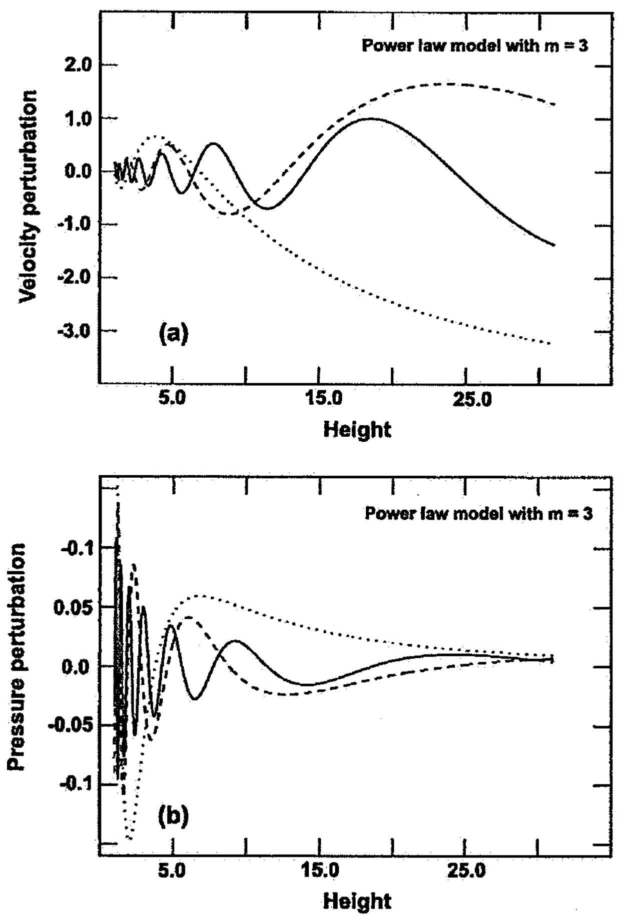

FIG. 3. Normalized velocity (a) and pressure (b) perturbations are plotted versus height in the nonisothermal model with the power law temperature distribution with $m=3$. The three considered cases correspond to $\omega=50\Omega_0$ (solid line), $\omega=25\Omega_0$ (dashed line), and $\omega=10\Omega_0$ (dotted line). In all these cases, the wave propagation begins at the height $z=z_0$.

state solutions of Euler's equation, which require that acoustic waves are propagating when $\omega>\Omega_0$ and nonpropagating when $\omega\leqslant\Omega_0$. However, there is no physical or mathematical reason to extrapolate these two conditions to the other models. Formally, the frequency Ω_0 appears in the arguments of all steady-state solutions, however, the results presented in Figs. 2–4 show that its influence on the behavior of the waves is rather minor. Therefore, it would be impossible to justify the choice of Ω_0 as a cutoff frequency for each model considered in this paper.

Another important observation is that Ω_0 does not explicitly depend on the temperature gradients but instead is related to z_0, which represents our choice of a fixed height in the nonisothermal models. Therefore, Ω_0 may become the cutoff for the model with $m=2$ but it cannot play the same role in the other models. Since the analytical solutions are not sufficient to determine the cutoff frequencies, we must develop another method that would allow us to derive the acoustic cutoff frequency for each model considered here.

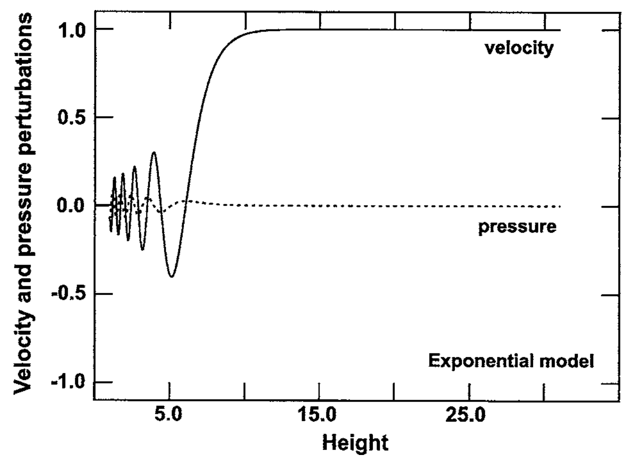

FIG. 4. Normalized velocity and pressure perturbations are plotted versus height in the nonisothermal model with the exponential temperature distribution. The frequency of acoustic waves is $\omega=50\Omega_0$ and the wave propagation begins at the height $z=z_0$.

IV. A METHOD TO DETERMINE CUTOFF FREQUENCIES

A. Klein-Gordon equations

Let us transform the acoustic wave equations [Eqs. (5)–(7)] into the corresponding Klein-Gordon equations [24]. We introduce the new variable $d\tau=dz/c_s$ and obtain

$$\frac{\partial^2 u_2}{\partial t^2}-\frac{\partial^2 u_2}{\partial \tau^2}+\left(\frac{c_s'}{c_s}\right)\frac{\partial u_2}{\partial \tau}=0, \tag{27}$$

$$\frac{\partial^2 p_2}{\partial t^2}-\frac{\partial^2 p_2}{\partial \tau^2}-\left(\frac{c_s'}{c_s}\right)\frac{\partial p_2}{\partial \tau}=0, \tag{28}$$

and

$$\frac{\partial^2 \rho_3}{\partial t^2}-\frac{\partial^2 \rho_3}{\partial \tau^2}-3\left(\frac{c_s'}{c_s}\right)\frac{\partial \rho_3}{\partial \tau}-2\left(\frac{c_s''}{c_s}\right)\rho_3=0, \tag{29}$$

where $c_s'=dc_s/d\tau$ and $c_s''=d^2c_s/d\tau^2$.

To remove the first order derivatives from these equations, we use $u_2(t,\tau)=\tilde{u}_2(t,\tau)e^{\zeta/2}, p_2(t,\tau)=\tilde{p}_2(t,\tau)e^{-\zeta/2}$ and $\rho_3(t,\tau)=\tilde{\rho}_3(t,\tau)e^{-3\zeta/2}$, where $\zeta=\int_0^\tau[c_s'(\tilde{\tau})/c_s(\tilde{\tau})]d\tilde{\tau}$. Thus, the transformed wave equations become the Klein-Gordon equations

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial \tau^2}+\Omega^2_{u,p,\rho}(\tau)\right][\tilde{u}_2(t,\tau),\tilde{p}_2(t,\tau),\tilde{\rho}_3(t,\tau)]=0, \tag{30}$$

where Ω_u,Ω_p, and Ω_p are known as the critical frequencies [25] given by

$$\Omega_u^2(\tau)=\Omega_\rho^2(\tau)=\frac{3}{4}\left(\frac{c_s'}{c_s}\right)^2-\frac{1}{2}\left(\frac{c_s''}{c_s}\right) \tag{31}$$

and

$$\Omega_p^2(\tau)=\frac{1}{2}\left(\frac{c_s''}{c_s}\right)-\frac{1}{4}\left(\frac{c_s'}{c_s}\right)^2. \tag{32}$$

An interesting result is that $\Omega_u^2=\Omega_\rho^2$, which means that the Klein-Gordon equations for the wave variables u_2 and ρ_3 are identical despite significantly different forms of Eqs. (27) and (29). Therefore, in the remaining parts of this paper, we shall only use the Klein-Gordon equation for $\tilde{u}_2$ and Ω_u.

B. Cutoff frequency for exponential model

In this model, $c_s(\xi)=c_{s0}e^{\xi/2}$ and $\tau=2a_0^{-1}e^{-\xi/2}$, where $a_0=c_{s0}/z_0$, and $c_s(\tau)=2z_0/\tau$, which gives $\Omega_u^2=-1/4\tau^2$ and $\Omega_p^2=3/4\tau^2$. After making the Fourier transform in time $[\tilde{u}^2(t,\tau),\tilde{p}^2(t,\tau)]=[\tilde{u}_3(\tau),\tilde{p}_3(\tau)]e^{-i\omega\tau}$ of Eq. (30), we obtain

$$\frac{d^2\tilde{u}_3}{d\tau^2}+\left(\omega^2+\frac{1}{4\tau^2}\right)\tilde{u}_3=0 \qquad (33)$$

and

$$\frac{d^2\tilde{p}_3}{d\tau^2}+\left(\omega^2-\frac{3}{4\tau^2}\right)\tilde{p}_3=0. \qquad (34)$$

Using the oscillation and turning point theorems (see Appendix B), the turning point frequencies $\Omega_{TP,u}$ and $\Omega_{TP,p}$ are calculated from the following conditions:

$$\Omega_{TP,u}^2+\frac{1}{4\tau^2}=\frac{1}{4\tau^2} \quad \text{and} \quad \Omega_{TP,p}^2-\frac{3}{4\tau^2}=\frac{1}{4\tau^2}, \qquad (35)$$

which give $\Omega_{TP,u}=0$, and $\Omega_{TP,p}=1/\tau$.

Having obtained the turning point frequencies, we now define the acoustic cutoff frequency $\Omega_{ac}(\tau)=\max[\Omega_{TP,u}(\tau),\Omega_{TP,p}(\tau)]$ and obtain

$$\Omega_{ac}(\tau)=\frac{1}{\tau} \quad \text{or} \quad \Omega_{ac}(\xi)=\Omega_0 e^{\xi}. \qquad (36)$$

The physical meaning of this cutoff frequency is that acoustic waves must have frequency ω higher than $\Omega_{ac}(\tau)$ in order to reach a given height τ and be propagating waves at this height. The cutoff has the same meaning in the ξ (or z) space. In other words, the acoustic cutoff frequency allows us to determine the height in the model at which waves of a given frequency become nonpropagating. This can be easily done by using the results presented in Fig. 5, which shows $\Omega_{ac}(\xi)$ for the exponential model (dotted line). If we assume that a wave source is located at z_0 in the model, then we may use $\Omega_{ac}(\xi)$ to determine the size of a region in nonisothermal media where acoustic waves of a given frequency are propagating. Since $\Omega_{ac}(\xi)\to\infty$ when $\xi\to\infty$, the higher the wave frequency the larger the height the wave can reach.

C. Cutoff frequencies for power law models

1. Models with $m\neq 2$

In these models, $c_s(\xi)=c_{s0}\xi^{m/2}$. To calculate the explicit form of the critical frequencies Ω_u and Ω_p, we first express c_s in terms of τ and obtain

$$c_s(\tau)=c_{s0}\left[\left(\frac{|2-m|}{2}\right)a_0\tau\right]^{m/(2-m)}, \qquad (37)$$

where $a_m=c_{s0}/z_0$. This gives

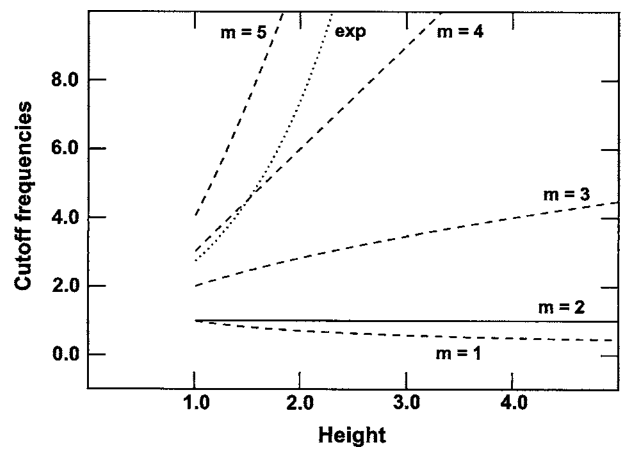

FIG. 5. The acoustic cutoff frequency Ω_{ac} normalized by Ω_0 is plotted as a function of height z/z_0 for the exponential model (exp) and the power law models with $m=1,2,3,4$, and 5; note that all the models start at $z=z_0$.

$$\Omega_{u,p}^2(\tau)=\frac{(4-m)m}{(2-m)^2}\frac{1}{4\tau^2}\equiv C_u^m\frac{1}{4\tau^2} \qquad (38)$$

and

$$\Omega_p^2(\tau)=\frac{(3m-4)m}{(2-m)^2}\frac{1}{4\tau^2}\equiv C_u^m\frac{1}{4\tau^2}. \qquad (39)$$

By making Fourier transforms in time $[\tilde{u}_2(t,\tau),\tilde{p}_2(t,\tau)]=[\tilde{u}_3(\tau),\tilde{p}_3(\tau)]e^{-i\omega\tau}$ of Eq. (30), we obtain

$$\left[\frac{d^2}{d\tau^2}+\left(\omega^2-\frac{[C_u^m,C_p^m]}{4\tau^2}\right)\right][\tilde{u}_3(\tau),\tilde{p}_3(\tau)]=0. \qquad (40)$$

The turning point frequencies $\Omega_{TP,u}$ and $\Omega_{TP,p}$ are calculated by using the oscillation and turning point theorems (see Appendix B). The following conditions are obtained:

$$[\Omega_{TP,u}^2,\Omega_{TP,p}^2]=\frac{1+[C_u^m,C_p^m]}{4\tau^2}, \qquad (41)$$

which give

$$\Omega_{TP,u}(\tau)=\frac{1}{|2-m|\tau} \quad \text{and} \quad \Omega_{TP,p}(\tau)=\frac{(1-m)}{(2-m)\tau}, \quad (42)$$

valid for all $m>0$ except $m=2$.

Having obtained the turning point frequencies $\Omega_{TP,u}$ and $\Omega_{TP,p}$, we now define the acoustic cutoff frequency $\Omega_{ac}(\tau)=\max[\Omega_{TP,u}(\tau),\Omega_{TP,p}(\tau)]$, and for each model with $m=1,3,4,5,6,\ldots$, the respective cutoff frequency becomes $\Omega_{ac}(\tau)=1/\tau, \Omega_{ac}(\tau)=2/\tau, \Omega_{ac}(\tau)=3/2\tau, \Omega_{ac}(\tau)=4/3\tau, \Omega_{ac}(\tau)=5/4\tau,\ldots$. This shows that the smallest and largest cutoff in τ space is found in the models with $m=1$ and $m=3$, respectively, and that for a given τ the values of the cutoffs for the models with $m\geqslant 4$ approach the smallest cutoff as $m\to\infty$. These are interesting results as they clearly indicate that if an acoustic wave of a given frequency ω reaches a height τ as propagating wave in the model with $m=3$, then this wave will also be propagating at the same height τ in all other models with $m\geqslant 4$.

Obviously, the τ space represents an useful mathematical description of the acoustic wave propagation in nonisothermal media. However, the waves propagate in the physical space, which is here the ξ or z space. For each model with $m=1,3,4,5,6,\ldots$, the respective cutoff frequency in ξ space becomes $\Omega_{\rm ac}(\xi)=\Omega_0\xi^{-1/2}, \Omega_{\rm ac}(\xi)=2\Omega_0\xi^{1/2}, \Omega_{\rm ac}(\xi)=3\Omega_0\xi, \Omega_{\rm ac}(\xi)=4\Omega_0\xi^{3/2}, \Omega_{\rm ac}(\xi)=5\Omega_0\xi^2,\ldots$. This sequence of the cutoff frequencies is plotted in Fig. 1, which shows that the propagating acoustic waves must have higher frequencies in order to reach the same height in the models with increasing values of m.

Our results also show that the acoustic cutoff frequency is defined by the turning point frequency $\Omega_{TP,p}$ for all models except the one with $m=1$. For the latter, $\Omega_{TP,p}=0$ and it is $\Omega_{TP,u}$ that defines the acoustic cutoff frequency. There are some smilarities between the model with $m=1$ and the exponential model as in both cases one of their turning point frequencies is zero; note that the steady-state solutions for these two models are also similar (see Sec. III). For all other models with $m>2$, both $\Omega_{TP,u}$ and $\Omega_{TP,p}$ are always nonzero, and $\Omega_{TP,p}=\Omega_{TP,u}$.

The physical meaning of the cutoff frequency $\Omega_{\rm ac}$ is the same as already described for the exponential model, namely, $\Omega_{\rm ac}$ allows us to determine the height in the nonisothermal medium at which waves of a given frequency become nonpropagating. In other words, the wave frequency ω must be larger than $\Omega_{\rm ac}(\tau)$ or $\Omega_{\rm ac}(\xi)$ for the waves to be propagating at a given height τ or ξ, which is consistent with the results given in Appendix C.

2. Model with *m*=2

In this case, $c_s(\xi)=c_{s0}\xi$ and the variables τ and z are related to each other by $\tau=a_0^{-1}\ln(\xi)$, with $a_0=c_{s0}/z_0$, and this gives $c_s(\tau)=c_{s0}e^{a_0\tau}$. Using Eqs. (31) and (32), we get $\Omega_u=\Omega_p=\Omega_p=\Omega_0$ and write Eq. (30) as

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial \tau^2}+\Omega_0^2\right)[\tilde{u}_2(t,\tau),\tilde{p}_2(t,\tau),\tilde{\rho}_3(t,\tau)]=0, \quad (43)$$

which shows that all three wave variables satisfy the same Klein-Gordon equation. Since Ω_0 is constant, we make Fourier transforms in time and space, and obtain the dispersion relation $(\omega^2-\Omega_0^2)=k_\tau^2$, where k_τ is a wave vector corresponding to τ. The fact that the dispersion relation can be derived in the τ space but not in the z space is an interesting, and rather surprising result.

The derived dispersion relation demonstrates that the acoustic waves are only propagating waves when $\omega>\Omega_0$, which means that the acoustic cutoff frequency $\Omega_{\rm ac}=\Omega_0$. It must be noted that in τ space the cutoff is a global quantity (the same in the entire medium) and its value is determined by a choice of the height z_0. Once z_0 is chosen, the values of T_{00} and c_{s0} can be determined and $\Omega_{\rm ac}$ can be calculated. The only condition required for propagating waves is that their frequencies are higher than the cutoff frequency. This result is consistent with that obtained in Sec. III.

TABLE I. The acoustic cutoff frequencies $\Omega_{\rm ac}$ derived for the nonisothermal media modeled with the exponential and power law temperature distributions.

Model	$c_s(\xi)$ [km/s]	Parameter m	$\Omega_{\rm ac}(\tau)$ [s^{-1}]	$\Omega_{\rm ac}(\xi)$ [s^{-1}]
Exponential	$c_{s0}e^{\xi/2}$	none	$1/\tau$	$\Omega_0 e^{\xi}$
Power law	$c_{s0}\xi^{m/2}$	0	none	none
		1	$1/\tau$	$\Omega_0/\xi^{1/2}$
		2	Ω_0	Ω_0
		3	$2/\tau$	$2\Omega_0\xi^{1/2}$
		4	$3/2\tau$	$3\Omega_0\xi^{2/2}$
		5	$4/3\tau$	$4\Omega_0\xi^{3/2}$
		6	$5/4\tau$	$5\Omega_0\xi^{4/2}$
		7	$6/5\tau$	$6\Omega_0\xi^{5/2}$
		8	$7/6\tau$	$7\Omega_0\xi^{6/2}$

D. Physical interpretation and applications

Our results allow us to introduce a new physical concept, namely, the local acoustic cutoff frequency that depends on spatial derivatives of the speed of sound and, as a result, vary with height in nonisothermal media. Clearly, the physical meaning of this cutoff is different from Lamb's global cutoff frequency described in Appendix A. To be more specific, the value of the local acoustic cutoff frequency at a given height determines the frequency that an acoustic wave must have in order to be propagating at this height. An important physical implication of this result is that any acoustic disturbance imposed on a nonisothermal medium would trigger a response at this local cutoff frequency. In other words, for a given medium that extends from, say, $\xi=1$ to $\xi=10$ with $\xi=z/z_0$, there is an interval of cutoff (natural) frequencies excited by the medium, and this interval can be uniquely determined when the range of ξ is given (see Table I). Obviously, the problem can also be reversed and the range of ξ can be determined once the interval of cutoff frequencies is known.

The results obtained in this paper can be experimentally verified. Nonisothermal media modeled by power law temperature distributions are designed for immediate use in laboratory experiments as they do not include external forces such as gravity and magnetic fields. Hence, the main results given in Table I and Fig. 5 can be verified by designing experiments in which the resulting interval of the cutoff frequencies for a given extent of a nonisothermal medium is measured.

Despite the simplicity of our models, one of the power law models can be used to describe the temperature variation in the solar transition region, where $p_0\approx$const and the temperature sharply increases with the atmospheric height [14]. We shall use the method developed in this paper to calculate the interval of cutoff frequencies in the solar transition region and compare these theoretical predictions with typical frequencies of atmospheric oscillations observed in this region. The results of this comparison will be presented in a separate paper.

Our method to determine the acoustic cutoff frequencies has been developed for nonisothermal media modeled with exponential and power law temperature distributions. In the following, we generalize the method so that it is applicable to nonisothermal media with arbitrary temperature gradients.

V. GENERALIZED METHOD

We begin with a general form of the wave equation describing acoustic waves propagating in a nonisothermal medium. Let us assume that there is a temperature gradient in the background medium in the z direction and consider the waves propagating solely in this direction. Then, the acoustic wave equation [2] can be written in the following general form:

$$\left[\hat{L}_{s,i}\left(\frac{\partial^2}{\partial t^2}, c_s^2\frac{\partial^2}{\partial z^2}, c_s\frac{dc_s}{dz}\frac{\partial}{\partial z}, c_s\frac{d^2c_s}{dz^2}\right)\right]\Phi_i = 0, \tag{44}$$

where $c_s(z)$ is the speed of sound and Φ_i, with $i=1$, 2, and 3, represents different wave variables which, for acoustic waves, are: the wave velocity, pressure and density perturbations. The fact that the acoustic wave operators $\hat{L}_{s,i}$ are usually different for different wave variables is shown here by the subscript i; as an example, see our results presented in Sec. II.

The acoustic wave equations given by Eq. (44) can be now transformed into Klein-Gordon equations [24,25]. This is done by using $d\tau = dz/c_s$ and removing the first derivative with $\Phi = \tilde{\Phi}e^{\kappa\zeta}$, where κ is a numerical factor whose value is different for different wave variables $\zeta = \int_0^\tau [c_s'(\tilde{\tau})/c_s(\tilde{\tau})]d\tilde{\tau}$ and $c_s' = dc_s/d\tilde{\tau}$. The result is

$$\left[\hat{L}_{\mathrm{KG},i}\left(\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial \tau^2}, \Omega_i^2(\tau)\right)\right]\tilde{\Phi}_i \equiv \left[\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \tau^2} + \Omega_i^2(\tau)\right]\tilde{\Phi}_i = 0, \tag{45}$$

where $\Omega_i = \Omega_i(c_s', c_s'')$, with $c_s' = dc_s/d\tau$ and $c_s'' = d^2c_s/d\tau^2$, are also known as the critical frequencies and their explicit forms are typically different for different wave variables [25].

The main advantage of transforming the acoustic wave equations into the Klein-Gordon equations is that the critical frequencies Ω_i become the only nonconstant coefficients in the equations, and that the different behavior of different wave variables is fully accounted for by differences between these frequencies. An interesting result is that the general form of the operators $\hat{L}_{\mathrm{KG},i}$ as given by Eq. (45), is the same for different wave variables [see Eqs. (27)–(29)] and it is also the same for different wave phenomena [25].

Having derived the Klein-Gordon equations, we then make Fourier transforms in time and obtain the steady-state Klein-Gordon equations for which the oscillation theorem can be used [26]. According to this theorem, one may compare the form of a given equation to another one whose solution is known (see Appendix B). In our method, we compare the steady-state Klein-Gordon equations to Euler's equation and determine frequencies that correspond to turning points of these equations; according to the results given in Appendix C, the turning points separate propagating and nonpropagating wave solutions. The comparison shows that the turning point frequencies $\Omega_{TP,i}$ are related to the critical frequencies Ω_i and that both $\Omega_i(\tau)$ and $\Omega_{TP,i}(\tau)$ are local quantities in nonisothermal media.

The final step of our method is to use $\Omega_{TP,i}(\tau)$ and determine the acoustic cutoff frequency $\Omega_{\mathrm{ac}}(\tau)$, which is achieved by taking $\Omega_{\mathrm{ac}}(\tau) = \max[\Omega_{TP,1}(\tau), \Omega_{TP,2}(\tau), \Omega_{TP,3}(\tau)]$. From a physical point of view, this choice can be justified by the fact that in order to have propagating acoustic waves at a height where the turning point is located, the wave frequency ω must always be higher than any turning point frequency. Since the location of the turning point is different for the waves of different frequencies, the waves of a given frequency ω are propagating at a certain height τ only if $\omega > \Omega_{\mathrm{ac}}(\tau)$, however, if $\omega \leq \Omega_{\mathrm{ac}}(\tau)$, the waves are nonpropagating. It is important to note that the acoustic cutoff frequency is uniquely determined by our method without formally solving either the wave equations or the Klein-Gordon equations. The generalized method presented here is robust and it can be applied to nonisothermal media of arbitrary temperature gradients. It is our hope that this generalized method will become an important tool in studies of acoustic waves in laboratory experiments and in realistic settings occuring in Nature.

VI. SUMMARY

The propagation of linear and adiabatic acoustic waves in non-isothermal media modeled with exponential and power law temperature distributions were studied herein. The acoustic wave equations were derived and analytical solutions of these equations were obtained. It was shown that, with exception of one specific model, the solutions were not sufficient to determine acoustic cutoff frequencies for all models. Because of this limitation, we developed a method that allows deriving the cutoff frequencies for all considered models.

Our method is based on transforming the acoustic wave equations for all wave variables into Klein-Gordon equations. The oscillation theorem is then applied to steady-state Klein-Gordon equations, and frequencies corresponding to the turning points are calculated for each wave variable. The acoustic cutoff frequency is identified with the largest turning point frequency. The choice is physically justified by the fact that in order to have propagating acoustic waves at a height where the turning point is located the wave frequency ω must always be higher than any turning point frequency. Since the location of the turning point is different for the waves of different frequencies, the waves of a given frequency ω are propagating at a certain height z only if $\omega > \Omega_{\mathrm{ac}}(z)$, otherwise, the waves are nonpropagating. It is important to note that the acoustic cutoff frequency is uniquely determined by our method without formally solving either the wave equations or the Klein-Gordon equations.

The most important results obtained in this paper are summarized in Table I, which shows the relationships between the cutoff frequencies in τ and z space, and in Fig. 5, which shows the dependence of the cutoffs on the height z/z_0 in each model. In both spaces, the acoustic cutoff frequency allows us to determine the height in the medium at which waves of a given frequency become nonpropagating; for the z space, this is shown in Fig. 5. If we assume that a wave source is located at z_0 in a given nonisothermal medium, then $\Omega_{ac}(Z)$ tells us what frequencies the acoustic waves must have in order to be propagating at a given height z. This shows that the derived acoustic cutoff frequencies are local quantities in nonisothermal media and that their physical meaning is significantly different than the global cutoff frequency first introduced by Lamb [8]. Finally, we discussed possible experimental and observational verifications of our results and presented a generalized method that can be used to calculate the acoustic cutoff frequencies for nonisothermal media with arbitrary temperature gradients.

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APPENDIX A: LAMB'S ACOUSTIC CUTOFF FREQUENCY

In his original work [8–10], Lamb considered acoustic waves propagating in the z direction in the background medium with the gravity $\vec{g}=-g\hat{z}$ and the density gradient $\rho_0(z)=\rho_{00}\exp(-z/2H)$, where ρ_{00} is the gas density at the height $z=0$ and $H=c_s^2\gamma g$ is the density scale height, with γ being the ratio of specific heats and c_s being the speed of sound. In his model, the background gas pressure p_0 varies with height z, however, the temperature T_0 remains constant. As a result, H=const and c_s=const. This stratified but otherwise isothermal medium is often referred to as an isothermal atmosphere because of its applications to the solar and stellar atmospheres [3,4,6].

The acoustic wave equations for the wave variables $u_1(t,z), p_1(t,z)$ and $\rho_1(t,z)$ become

$$\left[\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial z^2} + \Omega_{ac}^2\right](u_1,p_1,\rho_1) = 0 \qquad \text{(A1)}$$

where the acoustic cutoff freqency $\Omega_{ac}=c_s/2H$. Note that the form of the wave equation is the same for each wave variable.

Since Ω_{ac}=const, one can make Fourier transforms in time and space and derive the global dispersion relation $(\omega^2-\Omega_{ac}^2)=k^2c_s^2$, where ω is the wave frequency and $k=k_z$ is the wave vector. This shows that the waves are propagating when $\omega>\Omega_{ac}$ and k is real, and they are nonpropagating when either $\omega=\Omega_0$ with $k=0$ or $\omega<\Omega_0$ with k being imaginary; in the latter case, the waves are called evanescent waves.

APPENDIX B: OSCILLATION AND TURNING POINT THEOREMS

Oscillation theorem. Consider an ordinary differential equation of the form

$$\frac{d^2\phi}{dx^2} + \Phi(x)\phi = 0, \qquad \text{(B1)}$$

which is known to have all of its solutions to be periodic. Assume that there is another equation of the form

$$\frac{d^2\psi}{dx^2} + \Psi(x)\psi = 0, \qquad \text{(B2)}$$

where $\Psi(x)>\Phi(x)$ for all x. Then, all of the solutions of Eq. (B2) are also periodic. The proof of this powerful theorem that gives a condition for the existence of periodic solutions is simple and available in the literature [26].

Turning point theorem. Consider an ordinary differential equation of the form

$$\frac{d^2\phi}{dx^2} + \Phi(x)\phi = 0, \qquad \text{(B3)}$$

which is known to have a turning point that separates the periodic and nonperiodic solutions. Assume that there is another equation of the form

$$\frac{d^2\psi}{dx^2} + \Psi(x)\psi = 0. \qquad \text{(B4)}$$

This equation has a turning point only if the condition, $\Psi(x)=\Phi(x)$ is satisfied for all x. The proof is trivial since the condition turns Eq. (B4) into Eq. (B3).

APPENDIX C: EULER'S EQUATION AND ITS TURNING POINT

In general, Euler's equation can be written in the following form [21]:

$$\frac{d^2y}{dx^2} + \frac{C_E}{4x^2}y = 0, \qquad \text{(C1)}$$

where C_E is a constant whose value determines the form of the solution. For $C_E>1$, the equation has periodic solutions, however, the solutions become nonperiodic when $C_E<1$, and finally for $C_E=1$ there is a turning point, which separates these two distinct types of solutions.

A general form of steady-state Klein-Gordon equation [25] is

$$\frac{d^2Y_i}{dx^2} + (\omega^2 - \Omega_i^2)Y_i = 0, \qquad \text{(C2)}$$

where the form of the critical frequencies $\Omega_i^2(x)$, with $i=1,2$ and 3, may be different for different wave variables and for different models.

Using the oscillation theorem (see Appendix B), we show that the Klein-Gordon equation has periodic wave solutions when $[\omega^2-\Omega_i^2(x)]>1/4x^2$ is valid for all x. We use the turning point theorem (see Appendix B) to show that the Klein-Gordon equation has a turning point if, and only if, the condition $[\omega^2-\Omega_i^2(x)]=1/4x^2$ is satisfied for all x.

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